

Explicit Derivation of New Hyper-Kahler metric

Moulay Brahim Sedra

Université Ibn Tofail, Faculté des Sciences, Département de Physique, B.P. 133, 14000 Kénitra

Morocco

and

Université Mohammed V, Faculté des Sciences, Département de Physique, UFR Physique des

*Hautes Energies, B.P. 1400 Rabat, **Morocco****

(February 1, 2008)

Abstract

Using the harmonic superspace techniques in D=2 N=4, we present an explicit derivation of a new hyper-Kahler metric associated to the Toda like self interaction $H^{4+}(\omega, u) = (\frac{\xi^{++}}{\lambda})^2 \exp(2\lambda\omega)$. Some important features are also discussed.

*mailing address

I. INTRODUCTION

Recently, a particular interest has been devoted to the subject of hyper-Kahler metrics building which is solved in a nice way in the harmonic superspace [1,2] It is an important question of hyper-Kahler geometry, a subject much studied in modern theoretical physics, more especially in connection with the theory of gravitational instantons, moduli problems in monopole physics, string theory, and elsewhere [3–5].

An original contribution about hyper-Kahler geometry, where a complete geometrical characterization of manifolds for which extended supersymmetry is allowed was given in [3]. By requiring covariantly constant complex structures for the extended supersymmetry, the authors showed that there exist two possibilities given by $N = 2$ and $N = 4$ supersymmetry for which the Kahler and hyper-Kahler manifolds play a special role respectively. On the other hand, these hyper-Kahler structures are of great interest as they are involved recently in the problem of moduli spaces of monopoles. A more tractable example being the moduli space of BPS magnetic monopoles [6], which is shown to possess hyper-Kahler structure. Although the previous achievements, a lot of things remains to do. The classification of all complete, regular, hyper-Kahler manifolds remains an open question to this date, and even for some known examples, the explicit form of the metric has been difficult or impossible to determine so far.

Recall that there are only few examples that had been solved exactly, these are the Taub-Nut and the Eguchi-Hanson hyper-Kahler metrics exhibiting both a $U(2)=SU(2)\times U(1)$ isometries.

In the present work, we propose another steps toward the classification of metrics of hyper-Kahler geometry and compute explicitly the metric associated to the harmonic superspace Toda (Liouville) like self-interaction. Recall that the problem of finding the metrics amount to eliminating an infinite number of auxiliary fields. We derive a new metric which can be useful in the sense that the particular hyper-Kahler geometrical structure that it suspected to describe is connected to integrable models via the Toda like self interaction.

We start in section 2 by recalling some general properties of the hyper-Kahler metrics building program and present in sections 3 and 4 the details concerning the derivation of the metric. The explicit form of this metric is exposed in the appendix. We conclude in section 5 and discuss further the framework for possible applications.

II. GENERALITIES ON HYPER-KAHLER METRICS BUILDING FROM H.S

We start this section by recalling some general results of the hyper-Kahler metrics building from the harmonic superspace. The subject of hyper-Kahler metrics building is an interesting problem of hyper-Kahler geometry that can be solved in a nice way in harmonic superspace if one know how to solve the following non linear differential equations on the sphere S^2 [2]:

$$\partial^{++} q^+ - \partial^{++} \left[\frac{\partial V^{4+}}{\partial(\partial^{++} \bar{q}^+)} \right] + \frac{\partial V^{4+}}{\partial \bar{q}^+} = 0, \quad (2.1a)$$

$$\partial^{++} \bar{q}^+ + \partial^{++} \left[\frac{\partial V^{4+}}{\partial(\partial^{++} q^+)} \right] - \frac{\partial V^{4+}}{\partial q^+} = 0, \quad (2.1b)$$

where $q^+ = q^+(z, \bar{z}, u^\pm)$ and its conjugates $\bar{q}^+ = \bar{q}^+(z, \bar{z}, u^\pm)$ are complex fields defined on $C \times S^2$ respectively and are parametrized by the local analytic coordinates z, \bar{z} and the harmonic variable u^\pm . The symbol $\partial^{++} = u^{+i} \frac{\partial}{\partial u^{-i}}$ stands for the so-called harmonic derivative and $V^{4+} = V^{4+}(q, u)$ is an interacting potential depending in general on q^+, \bar{q}^+ , their derivatives and the u^\pm 's. The fields q^+ and \bar{q}^+ are globally defined on the sphere $S^2 = SU(2)/U(1)$ and may be expanded into an infinite series in power of harmonic variables (for the bosonic part) preserving the total U(1) charge in each term of the expansion as given here below

$$q^+(z, \bar{z}, u) = u_i^+ f^i(z, \bar{z}) + u_i^+ u_j^+ u_k^- f^{(ijk)}(z, \bar{z}) + \dots \quad (2.2a)$$

Note here that Eqs.(2.1), which fix the u -dependence of the q^+ 's is in fact the pure bosonic projection of a two dimensional $N = 4$ supersymmetric HS superfield equation of motion.

The remaining equations carry the spinor contributions and are shown to describe among others, the space time dynamics of the physical degrees of freedom namely, the four bosons $f^i(z, \bar{z})$, $\bar{f}^i(z, \bar{z})$, $i = 1, 2$, and their $D = 2$ $N = 4$ supersymmetric partners.

An equivalent approach to write Eq.(2.1) is to use the How-Stelle-Townsend (HST) realization [7]

$$\partial^{++2}\omega - \partial^{++}[\frac{\partial H^{4+}}{\partial(\partial^{++}\omega)}] + \frac{\partial H^{4+}}{\partial\omega} = 0, \quad (2.3a)$$

where $\omega = \omega(z, \bar{z}, u)$ is a real field with zero $U(1)$ charge defined on $\mathbb{C} \times S^2$ and with its leading terms of its harmonic expansion given by

$$\omega(z, \bar{z}, u) = u_i^+ u_j^- f^{ij}(z, \bar{z}) + u_i^+ u_j^+ u_k^- u_l^- g^{(ijkl)}(z, \bar{z}) + \dots \quad (2.4)$$

Similarly as in Eq.(2.1), the interaction potential H^{4+} depends in general on ω , its derivatives and the harmonics. Note the important observation of [8], that one can always pass from the q^+ hypermultiplet to the ω hypermultiplet via a duality transformation [9] by making a change of variables.

In the remarkable case where the potentials H^+ and V^{4+} do not depend on the derivatives of the fields q^+ and ω , Eqs.(1) and (3) reduce then to

$$\partial^{++}q^+ + \frac{\partial V^{4+}}{\partial \bar{q}^+} = 0 \quad (2.5a)$$

$$\partial^{++2}\omega + \frac{\partial H^{4+}}{\partial\omega} = 0 \quad (2.5b)$$

As the solutions of these equations depend naturally on the potentials V^{4+} and H^{4+} , the finding of these solutions is not an easy question. There are only few examples that had been solved exactly. The first example is the Taub-Nut metric of the four-dimensional Euclidean gravity. The potential V^{4+} of this model is given by

$$V^{4+}(q^+, \bar{q}^+) = \frac{\lambda}{2} q^+ \bar{q}^{+2} \quad (2.6)$$

where λ is a real coupling constant. For this potential, the equation of motion reads

$$\partial^+ q^+ + \lambda q^+ (\bar{q})^+ \bar{q}^+ = 0 \quad (2.7)$$

and its solution is given by

$$q^+(z, \bar{z}, u) = u_i^+ f^i(z, \bar{z}) \exp(-\lambda u_k^+ u_l^- f^{(k} f^{l)}) \quad (2.8)$$

Note that the knowledge of this solution is an important steps in the identification of the metric of the manifold parametrized by the bosonic fields $f^i(z, \bar{z})$ and $\bar{f}^i(z, \bar{z})$ of the $D = 2$ $N = 4$ supersymmetric non linear Taub-Nut σ -model.

The second example we consider, is the Eguchi-Hanson model. This model had been also solved exactly and correspond to the following potential

$$H^{4+}(\omega) = \left(\frac{\xi^{++}}{\omega}\right)^2 \quad (2.9)$$

where the dimensionless quantity ξ^{++} is given by

$$\xi^{++} = \xi^{ij} u_i^+ u_j^+ \quad (2.10)$$

in terms of the real isovector coupling constant ξ^{ij} . Thus unlike the TN action, the EH action contains explicit harmonics. Details concerning these two models can be found in the following refs. [2].

Recently, a new integrable model had been proposed [10]. This model was obtained by focusing on Eq.(2.5b) and looking for potentials leading to exact solutions of this equation. The method used in this issue consist in wondering plausible integrable equations by proceeding as formal analogy with the known integrable two-dimensional non linear differential equations especially the Liouville equation and its Toda generalizations [11]. The important result in this sense was the proposition of the following potential

$$H^{4+}(\omega, u) = \left(\frac{\xi^{++}}{\lambda}\right)^2 \exp(2\lambda\omega) \quad (2.11)$$

which leads via Eq.(2.5b) to the following non linear differential equation of motion

$$\lambda(\partial^{++})^2 \omega - (\xi^{++})^2 \exp(2\lambda\omega) \quad (2.12)$$

Using the formal analogy with the $SU(2)$ Liouville($SU(N)$ Toda) equation, we showed that this equation (2.12) is integrable. The explicit solution of this non linear differential equation reads

$$(\xi^{++})^2 \exp(\lambda\omega) = \frac{u_i^+ u_j^+ f^{ij}(z, \bar{z})}{1 - u_k^+ u_l^- f^{kl}(z, \bar{z})} \quad (2.13)$$

Furthermore, the origin of the integrability in Eq.(2.12) is shown to deal with the existence of a symmetry (conformal symmetry) generated by the following conserved current

$$T^{4+} = \partial^{++2} - \frac{1}{\lambda} \partial^{++2} \omega \quad (2.14)$$

with $\partial^{++} T^{4+} = 0$

III. COMPUTATION OF THE BOSONIC METRIC

We focus in this section to apply the general method presented in ref. [2] to derive the hyper-Kahler metric associated to the proposed potential Eq.(2.11) using the harmonic superspace approach. This method consist in writing the action describing the general coupling of the analytic superfield Ω we are interested in and in deriving the corresponding equation of motion. For this action which correspond to some hyper-Kahler manifold, one has only to expand the equations of motion in spinor coordinates θ , omit the fermions and solve the auxiliary fields equations. Substituting the solution into the original action and integrating over the harmonics ones u^\pm , yields the required component form of the action from which the hyper-Kahler metric can be obtained. Let us first recall that the harmonic superspace (HS) is parametrized by the supercoordinates $Z^M = (z_A^M, \theta_r^-, \bar{\theta}_r^-)$ where $z_A^M = (z, \bar{z}, \theta_r^+, \bar{\theta}_r^+, u^\pm)$ are the supercoordinates of of the so-called analytic subspace in which $D = 2$ $N = 4$ supersymmetric theories are formulated. The integral measure of the harmonic superspace is given in the Z_A^M basis by $d^2 z d^4 \theta^+ du$. The matter superfield $(O^4, (\frac{1}{2})^4)$ is realized by two dual analytic superfields $Q^+ = Q^+(z, \bar{z}, \theta^+, (\bar{\theta})^+, u)$ and $\Omega = \Omega(z, \bar{z}, \theta^+, (\bar{\theta})^+, u)$ whose leading bosonic fields are respectively given by q^+ and ω Eqs.(2.2a) and (2.4). The model

we are interested in and which describe the coupling of the analytic superfield Ω is given by the following action [10]

$$S[\Omega] = \int d^2z d^4\theta^+ du \left\{ \frac{1}{2} (D^{++}\Omega)^2 + \frac{1}{2} \left(\frac{\xi^{++}}{\lambda} \right)^2 e^{2\lambda\Omega} \right\} \quad (3.1)$$

where λ is the coupling constant of the model and $\xi^{++} = u_i^+ u_j^+ \xi^{(ij)}$ a constant isotriplet similar to that appearing in the Eguchi-Hanson model and where D^{++} is the harmonic derivative given by

$$D^{++} = \partial^{++} - 2\bar{\theta}_r^+ \theta_r^+ \partial_{-2r} \quad (3.2)$$

The equation of motion corresponding to this action Eq.(3.1) reads

$$\lambda D^{++2}\Omega - \xi^{++2} e^{2\lambda\Omega} = 0 \quad (3.3)$$

where Ω is the analytic superfield which expand in θ_r^+ and $\bar{\theta}_r^+$ series as

$$\begin{aligned} \Omega = & \omega + [\theta_-^+ \theta_+^+ F^{--} + \bar{\theta}_+^+ \bar{\theta}_-^+ \bar{F}^{--}] \\ & + [\bar{\theta}_-^+ \theta_+^+ G^{--} + \bar{\theta}_+^+ \theta_-^+ \bar{G}^{--}] \\ & + [\bar{\theta}_-^+ \theta_-^+ B_{++}^{--} + \bar{\theta}_+^+ \theta_+^+ B_{--}^{--}] \\ & + [\bar{\theta}_-^+ \theta_-^+ \bar{\theta}_+^+ \theta_+^+ \Delta^{-4}] \end{aligned} \quad (3.4)$$

Substituting Eq.(3.4) into Eq.(3.3), one obtains the following non linear differential equations

$$\lambda \partial^{++2}\omega - \xi^{++2} e^{2\lambda\omega} = 0 \quad (3.5)$$

$$\partial^{++2}F^{--} - 2\xi^{++2}F^{--}e^{2\lambda\omega} = 0 \quad (3.6a)$$

$$\partial^{++2}\bar{F}^{--} - 2\xi^{++2}\bar{F}^{--}e^{2\lambda\omega} = 0 \quad (3.6b)$$

$$\partial^{++2}G^{--} - 2\xi^{++2}G^{--}e^{2\lambda\omega} = 0 \quad (3.7a)$$

$$\partial^{++2}\bar{G}^{--} - 2\xi^{++2}\bar{G}^{--}e^{2\lambda\omega} = 0 \quad (3.7b)$$

$$\partial^{++2}B_{++}^{--} - 2\xi^{++2}B_{++}^{--}e^{2\lambda\omega} = 4\partial^{++}\partial_{++}\omega \quad (3.8a)$$

$$\partial^{++2}B_{--}^{--} - 2\xi^{++2}B_{--}^{--}e^{2\lambda\omega} = 4\partial^{++}\partial_{--}\omega \quad (3.8b)$$

$$\begin{aligned} \partial^{++2}\Delta^{(-4)} - 4\partial^{++}\partial_{--}B_{++}^{--} - 4\partial^{++}\partial_{++}B_{--}^{--} \\ = 2\xi^{++2}e^{2\lambda\omega}[\Delta^{(-4)} + 2\lambda(F^{--}\bar{F}^{--} - G^{--}\bar{G}^{--} + B_{++}^{--}B_{--}^{--})] \end{aligned} \quad (3.9)$$

The Liouville-like equation of motion Eq.(3.5), is a constraint equation fixing the dependence of ω in terms of the physical bosonic fields $f^{(ij)}$ of the $D = 2$ $N = 4$ hypermultiplet. The knowledge of the solution of this nonlinear equation is necessary as it is one of the main crucial steps in this program. The second set of relations Eqs.(3.6-8) describe the equations of motion of the auxiliary fields F^{--} , G^{--} and B_{rr}^{--} of canonical dimension one. The last equation Eq.(3.9) gives the equation of motion of the Lagrange field $\Delta^{(-4)}$ of canonical dimension two in terms of ω and the other auxiliary fields. To solve these equations of motion, one start first by solving the Liouville-like equation of motion Eq.(3.5) whose solution[11], originated from integrability and conformal symmetry in two dimensions, reads in the HS language as [10]

$$\xi^{++}e^{\lambda\omega} = \frac{f^{++}}{1-f} \quad (3.10)$$

Next, we integrate the action Eq.(3.1) with respect to the Grassmann variables θ . One find the following result

$$\begin{aligned} S = \int d^2z du \{ [\partial^{++}\omega\partial^{++}\Delta^{(-4)} + \frac{1}{\lambda}\xi^{++2}e^{2\lambda\omega}\Delta^{(-4)}] \\ + [\partial^{++}F^{--}\partial^{++}\bar{F}^{--} + 2\xi^{++2}e^{2\lambda\omega}F^{--}\bar{F}^{--}] \\ - [\partial^{++}G^{--}\partial^{++}\bar{G}^{--} + 2\xi^{++2}e^{2\lambda\omega}G^{--}\bar{G}^{--}] \\ + [\partial^{++}B_{--}^{--}\partial^{++}B_{++}^{--} - 2\partial^{++}\omega(\partial_{--}B_{++}^{--} + \partial_{++}B_{--}^{--})] \\ + [\partial^{++}B_{--}^{--}\partial^{++}B_{++}^{--} - 2(\partial_{++}\omega\partial^{++}B_{--}^{--} + \partial_{--}\omega\partial^{++}B_{++}^{--}) \\ + 2\xi^{++2}e^{2\lambda\omega}B_{--}^{--}B_{++}^{--}] \} \end{aligned} \quad (3.11)$$

Using the equation of motion for ω Eq.(3.5), one show easily that the following harmonic integrals are vanishing

$$\int du [\partial^{++}\omega \partial^{++}\Delta^{(-4)} + \frac{1}{\lambda}\xi^{++2}e^{2\lambda\omega}\Delta^{(-4)}] = 0 \quad (3.12)$$

$$\int du [\partial^{++}F^{--}\partial^{++}\bar{F}^{--} + 2\xi^{++2}e^{2\lambda\omega}F^{--}\bar{F}^{--}] = 0 \quad (3.13)$$

$$\int du [\partial^{++}G^{--}\partial^{++}\bar{G}^{--} + 2\xi^{++2}e^{2\lambda\omega}G^{--}\bar{G}^{--}] = 0 \quad (3.14)$$

$$\int du [\partial^{++}B_{--}^{--}\partial^{++}B_{++}^{--} - 2(\partial_{++}\omega\partial^{++}B_{--}^{--} + \partial_{--}\omega\partial^{++}B_{++}^{--})] \quad (3.15)$$

$$+ 2\xi^{++2}e^{2\lambda\omega}B_{--}^{--}B_{++}^{--}] = 0 \quad (3.16)$$

These vanishing integrals serve to eliminate the auxiliary fields F, G and Δ . The resulting bosonic action is then

$$S = \int d^2z du \{ 4\partial_{++}\omega\partial^{++}B_{--}^{--} - 2\partial^{++}\omega(\partial_{--}B_{++}^{--} + \partial_{++}B_{--}^{--}) - 2\lambda\partial^{++2}\omega B_{--}^{--}B_{++}^{--} \} \quad (3.17)$$

In order to obtain the purely bosonic theory, one have to reduce more this action which depends now only on the fields B and ω . To do this, one needs to solve the differential equation Eq.(3.8) for the auxiliary fields $B_{rr}^{--}, r = \pm$, namely

$$\partial^{++2}B_{rr}^{--} = 2\xi^{++2}e^{2\lambda\omega}B_{rr}^{--} + 4\partial^{++}\partial_{rr}\omega, \quad (3.18)$$

this equation gives the relation between B_{rr}^{--} and the fields ω . Finding a solution for this equation is not an easy exercise, because the behavior of B_{rr}^{--} seems to be non linear in term of ω . But, remarking just the fact that when ω is zero or a space-time constant, a particular solution of B_{rr}^{--} is zero. One propose then a solution of Eq.(3.18) of the form:

$$B_{rr}^{--} = \xi^{--}\partial_{rr}\omega, \quad (3.19)$$

where $\xi^{--} = u_i^- u_j^- \xi^{ij}$ is an $SU(2)$ triplet constant for which one can set $\partial^{++}\xi^{--} = 2$.

Injecting this solution into the action Eq.(3.17) one obtains

$$S = \int d^2z du \{ \partial_{++}\partial_{--}\omega [-2\lambda\xi^{--2}\partial^{++2}\omega + 4\lambda\xi^{--}\partial^{++}\omega + 8] - 4\xi^{--}\partial^{++}\omega\partial_{++}\partial_{--}\omega \} \quad (3.20)$$

showing the dependence of the action only on the bosonic degrees of freedom $f = u_i^+ u_j^- f^{ij}$ defined in Eq.(3.10). What remains to do now is to use the equation of motion Eq.(3.10) for ω and integrate over the harmonics u to derive the purely bosonic action from which one can easily identify the metric associated to our hyper-Kahler potential

$$H^{4+}(\Omega, u) = \left(\frac{\xi^{++}}{\lambda}\right)^2 \exp(2\lambda\Omega) \quad (3.21)$$

IV. FINDING THE METRIC

We start from the action Eq.(3.20) and consider the solution Eq.(3.10) which imply

$$\partial_{rr}\omega = \frac{1}{\lambda} \left[\frac{\partial_{rr}f^{++}}{f^{++}} + \frac{\partial_{rr}f}{1-f} \right] \quad (4.1)$$

Injecting this formal expression into the bosonic action Eq.(3.20), one obtains the following result

$$\begin{aligned} S[f] = \int d^2z du \frac{1}{\lambda^2} \{ & 8 \left(\frac{\partial_{--}f^{++}\partial_{++}f^{++}}{(f^{++})^2} + \frac{\partial_{++}f^{++}\partial_{--}f}{f^{++}(1-f)} \right. \\ & + \frac{\partial_{--}f^{++}\partial_{++}f}{f^{++}(1-f)} + \frac{\partial_{--}f\partial_{++}f}{(1-f)^2} \Big) \\ & + 4\xi^{--} \left(2 \frac{\partial_{--}f^{++}\partial_{++}f^{++}}{f^{++}(1-f)} - \frac{\partial_{--}\partial_{++}f^{++}}{1-f} \right. \\ & + \frac{\partial_{++}f^{++}\partial_{--}f + \partial_{++}f\partial_{--}f^{++} + f^{++}\partial_{++}\partial_{--}f}{(1-f)^2} \Big) \\ & - 2\xi^{--2} \left(\frac{\partial_{--}f^{++}\partial_{++}f^{++}}{(1-f)^2} + f^{++} \left[\frac{\partial_{--}f\partial_{++}f^{++}}{(1-f)^3} + \frac{\partial_{++}f\partial_{--}f^{++}}{(1-f)^3} \right] \right. \\ & \left. \left. + f^{++2} \frac{\partial_{++}f\partial_{--}f}{(1-f)^4} \right) \right\} \end{aligned} \quad (4.2)$$

where

$$f = u_i^+ u_j^- f^{(ij)} + f^0 \quad (4.3a)$$

$$f^{++} = u_i^+ u_j^+ f^{(ij)} \quad (4.3b)$$

$$\xi^{--} = u_i^- u_j^+ \xi^{(ij)} \quad (4.3c)$$

and where the symbol (ij) stands for the symmetric part of the tensor indices i, j . The antisymmetric part f^{++} is omitted as it leads simply to constant terms. Note that the obtained action Eq.(4.2) contains terms with singularity around $f = u^{+i} u_i^-$. This singularity is originated from the solution of Toda-(Liouville)like equation of motion Eq.(3.10). As it is worth stressing that not every solution of the continual Toda(Liouville) equation of motion has a good space time interpretation, since the corresponding metric might be incomplete with singularities, it seems at first sight that our hyper-Kahler metric will be incomplete. But using some algebraic manipulations, one can derive the complete form of the metric inspired from the first leading terms in a nice way. To derive the metric from the bosonic action Eq.(4.2), one use steps by steps the following operations. Starting once again from Eq.(4.2) and considering the following approximation

$$\frac{1}{(1-f)^\epsilon} = \sum_{i=0}^{\infty} \frac{(\epsilon+i-1)!}{(\epsilon-1)! i!} f^i, \epsilon = 1, 2, 3 \dots \quad (4.4)$$

with $f = u_i^+ u_j^- f^{(ij)} + f^0$. Injecting this expression into Eq.(4.2) and integrating over the harmonics once the power $f^i(z, \bar{z})$ of the bosonic field f in Eq.(4.4) are expressed as a series in terms of $f^{(ij)}, f^0$ and the symmetrized product of harmonics. To do this, one have also to use the standard reduction identities [2]

$$u_i^+ u_{(j_1}^+ \dots u_{j_n}^+ u_{k_1}^- \dots u_{k_m}^- = u_i^+ u_{(j_1}^+ \dots u_{k_m}^- + \frac{m}{m+n+1} \varepsilon_{i(k_1} u_{j_1}^+ \dots u_{j_n}^+ u_{k_2}^- \dots u_{k_m}^- \quad (4.5a)$$

$$u_i^- u_{(j_1}^+ \dots u_{j_n}^+ u_{k_1}^- \dots u_{k_m}^- = u_i^- u_{(j_1}^+ \dots u_{k_m}^- - \frac{n}{m+n+1} \varepsilon_{i(j_1} u_{j_2}^+ \dots u_{j_n}^+ u_{k_2}^- \dots u_{k_m}^- \quad (4.5b)$$

and the u_i^\pm integration rules

$$\int du (u^+)^m (u^-)^n (u^+)_{(k} (u^-)_{l)} = \begin{cases} \frac{(-1)^n m! n!}{(m+n+1)!} \delta_{(j_1}^{(i_1} \dots \delta_{j_{k+l)}^{i_{m+n})} & \text{if } m = l \text{ and } n = k \\ 0 & \text{otherwise} \end{cases} \quad (4.6)$$

with

$$(u^+)^m (u^-)^n \equiv u^{+(i_1} \dots u^{+i_m} u^{-j_1} u^{-j_n)} \quad (4.7)$$

To finally integrate over the harmonics, one learn from the content of the action Eq.(4.2) in u^\pm , that we should use the approximation Eq.(4.4) at least up to the six order in ε . Lengthy and very hard calculations leads finally to the following purely bosonic action

$$\begin{aligned}
S_{bos}[f] = \int d^2z \frac{1}{\lambda^2} \{ & A_{ijkl} \partial_{++} f^{ij} \partial_{--} f^{kl} \\
& + B_{ij} [\partial_{++} f^0 \partial_{--} f^{ij} + \partial_{++} f^{ij} \partial_{--} f^0] \\
& + C_{ij} \partial_{++} \partial_{--} f^{ij} + D \partial_{++} \partial_{--} f^0 \\
& + E \partial_{++} f^0 \partial_{--} f^0 + F \partial_{++} f^{ij} \partial_{--} f^{ij} \}
\end{aligned} \tag{4.8}$$

from which one can easily derive the metric. The tensor components of this metric are A_{ijkl} , B_{ij} , C_{ij} , D , E , and F such that

$$\begin{aligned}
A_{ijkl} &= A_{jikl} = A_{ijlk} \\
B_{ij} &= B_{ji}
\end{aligned} \tag{4.9}$$

$$C_{ij} = C_{ji} \tag{4.10}$$

The first explicit expression obtained for this metric is of course incomplete due the previous approximation. The missing terms in this metric are easily recuperated by looking just at the behavior of the first leading terms of the components A_{ijkl} , B_{ij} , C_{ij} , D , E , and F . We present in the appendix, the complete expression of the metric described by the bosonic field f . From the purely bosonic action Eq.(4.8), one can read lot of properties. Note for the moment only the following. The constant λ is just the coupling constant of the Liouville-(Toda)like theory. The components of the bosonic (physical) field f are dimensionless and are interpreted as the internal coordinates of the hyper-Kahler manifold associated to the proposed potential Eq.(2.11). Since the metric $(A_{ijkl}, B_{ij}, C_{ij}, D, E, F)$ is dimensionless and does not explicitly depend on the coupling λ , one have to pass from the $f = (f^0, f^{ij})$ to the true physical bosonic field by performing the following rescaling $f = \lambda f_{phys}$.

V. CONCLUSION

We have derived explicitly the metric associated to the $SU(2)$ Toda-like hyper-Kahler Potential. The idea to work this metric was first introduced in [10]. But only recently when studying the Witten-article [12] (where the author explain that one of the reason to go to eleven dimensional M-theory is that the study of sixbranes, described by the multi-Taub-Nut hyper-Kahler metric, become more simple) we asked the question what will be the importance of our metric in this context? This and others important questions which can help to achieve the old problem of classifying all the Hyper-Kahler manifolds will be discussed in future works.

ACKNOWLEDGEMENT

I would like to thank I. Bandos, K.S. Narain, M. O'loughlin and E.H. Saidi for valuable discussions. I would like to thank Professor M. Virasoro for good hospitality at ICTP where this work was done. I acknowledge also Professor S. Randjbar-Daemi and the High Energy section for scientific help.

APPENDIX:

$$\begin{aligned}
A_{ijkl} = & \frac{16}{\lambda^2} \frac{1}{f^{(ij)} f^{(kl)}} - \frac{168}{\lambda^2} \xi^{eh} \xi^{mn} f^{qr} f^{st} f_{(eh} f_{mn} f_{qr} f_{st} f_{ij} f_{kl}) \\
& + \frac{8}{30\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+3)!}{2N!} f^{0N} + \sum_{n=1}^{\infty} A_1(N, n) f^{0N-2n} (ff)^n \right\} f_{(ij} f_{kl)} \\
& - \frac{4}{3\lambda^2} \sum_{N=0}^{\infty} \left\{ (N+1) f^{0N} + \sum_{n=1}^{\infty} A_2(N, n) f^{0N-2n} (ff)^n \right\} \left(\frac{f_{(ij)}}{f^{(kl)}} + \frac{f_{(kl)}}{f^{(ij)}} \right) \\
& - \frac{14}{15\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+2)!}{2N!} f^{0N} + \sum_{n=1}^{\infty} A_3(N, n) f^{0N-2n} (ff)^n \right\} \\
& \times \xi^{mn} \left(\frac{f_{(ij} f_{mn)}}{f^{(kl)}} + \frac{f_{(mn} f_{kl)}}{f^{(ij)}} \right) \\
& + \frac{16}{3\lambda^2} \sum_{N=0}^{\infty} \left\{ f^{0N} + \sum_{n=1}^{\infty} A_4(N, n) f^{0N-2n} (ff)^n \right\} \left(\frac{\xi_{(ij)}}{f^{(kl)}} + \frac{\xi_{(kl)}}{f^{(ij)}} \right) \\
& - \frac{2}{35\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+4)!}{6N!} f^{0N} + \sum_{n=1}^{\infty} A_5(N, n) f^{0N-2n} (ff)^n \right\} \\
& \times \xi^{mn} f_{(mn} f_{ij} f_{kl)} \\
& - \frac{4}{9\lambda^2} \sum_{N=0}^{\infty} \left\{ (N+1)(N+2) f^{0N} + \sum_{n=1}^{\infty} A_6(N, n) f^{0N-2n} (ff)^n \right\} (\xi_{(ij)} f_{(kl)} + \xi_{(kl)} f_{(ij)}) \\
& - \frac{1}{63\lambda^2} \xi^{mn} \xi^{rs} f_{(ij} f_{kl} f_{mn} f_{rs}) \\
& - \frac{2}{45\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+3)!}{2N!} f^{0N} + \sum_{n=1}^{\infty} A_7(N, n) f^{0N-2n} (ff)^n \right\} \\
& \times \xi^{mn} (\xi_{(ij)} f_{(mn} f_{kl)} + \xi_{(kl)} f_{(mn} f_{ij)}) \\
& - \frac{8}{9\lambda^2} \sum_{N=0}^{\infty} \left\{ (N+1) f^{0N} + \sum_{n=1}^{\infty} A_8(N, n) f^{0N-2n} (ff)^n \right\} \\
& \times \xi_{(mn)} \xi_{(kl)} \\
& - \frac{4}{135\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+1)!}{12N!} f^{0N} + \sum_{n=1}^{\infty} A_9(N, n) f^{0N-2n} (ff)^n \right\} \\
& \times (\xi f)^2 f_{(ij} f_{kl)} \\
& - \frac{1}{945\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+7)!}{36N!} f^{0N} + \sum_{n=1}^{\infty} A_{10}(N, n) f^{0N-2n} (ff)^n \right\} \\
& \times ((\xi^f) + \xi^{mn} f^{rs} f_{(mn} f_{rs} f_{ij} f_{kl)}) \\
& + \frac{1}{693\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+7)!}{240N!} f^{0N} + \sum_{n=1}^{\infty} A_{11}(N, n) f^{0N-2n} (ff)^n \right\} \\
& \times \xi^{he} \xi^{mn} f^{rs} f_{(he} f_{mn} f_{rs} f_{ij} f_{kl)}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{105\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+5)!}{12N!} f^{0N} + \sum_{n=1} A_{12}(N, n) f^{0N-2n} (ff)^n \right\} \\
& \times \left\{ \xi^{mn} f^{rs} (f_{(mn} f_{rs} f_{kl}) \xi_{(ij)} + f_{(mn} f_{rs} f_{ij}) \xi_{(kl)}) + 2(\xi^f) \xi^{mn} f_{(mn} f_{ij} f_{kl)} \right\} \\
& + \frac{2}{27\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+3)!}{N!} f^{0N} + \sum_{n=1} A_{13}(N, n) f^{0N-2n} (ff)^n \right\} \\
& \times (\xi^f) (\xi_{(ij)} f_{(kl)} + \xi_{(kl)} f_{(ij)})
\end{aligned}$$

$$\begin{aligned}
B_{ij} = & -\frac{4}{3\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+2)!}{N!} f^{0N} + \sum_{n=1} B_1(N, n) f^{0N-2n} (ff)^n \right\} f_{(ij)} \\
& + \frac{8}{\lambda^2} \sum_{N=0}^{\infty} \left\{ f^{0N} + \sum_{n=1} B_2(N, n) f^{0N-2n} (ff)^n \right\} \frac{1}{f_{(ij)}} \\
& + \frac{1}{15\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+3)!}{N!} f^{0N} + \sum_{n=1} B_3(N, n) f^{0N-2n} (ff)^n \right\} \xi^{mn} f_{(mn} f_{ij)} \\
& + \frac{8}{3\lambda^2} \sum_{N=0}^{\infty} \left\{ (N+1) f^{0N} + \sum_{n=1} B_4(N, n) f^{0N-2n} (ff)^n \right\} \xi_{(ij)} \\
& + \frac{1}{315\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+8)!}{6!N!} f^{0N} + \sum_{n=1} B_5(N, n) f^{0N-2n} (ff)^n \right\} \\
& \times \xi^{eh} \xi^{mn} f^{rs} f^{tv} f_{(eh} f_{mn} f_{rs} f_{tv} f_{ij)} \\
& + \frac{2}{27\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+4)!}{3N!} f^{0N} + \sum_{n=1} B_6(N, n) f^{0N-2n} (ff)^n \right\} (\xi f)(\xi f) f_{ij} \\
& + \frac{1}{105\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+6)!}{18N!} f^{0N} + \sum_{n=1} B_7(N, n) f^{0N-2n} (ff)^n \right\} (\xi f) \xi^{mn} f^{rs} f_{(mn} f_{rs} f_{ij)} \\
& - \frac{1}{315\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+6)!}{48N!} f^{0N} + \sum_{n=1} B_8(N, n) f^{0N-2n} (ff)^n \right\} \xi^{eh} \xi^{mn} f^{rs} f_{(eh} f_{mn} f_{rs} f_{ij)} \\
& - \frac{1}{45\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+4)!}{2N!} f^{0N} + \sum_{n=1} B_9(N, n) f^{0N-2n} (ff)^n \right\} \\
& \times (\xi_{ij} \xi^{(mn} f^{rs)} f_{(mn} f_{rs)} + (\xi f) \xi^{mn} f_{(ij} f_{mn)}) \\
& - \frac{4}{9\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+2)!}{N!} f^{0N} + \sum_{n=1} B_{10}(N, n) f^{0N-2n} (ff)^n \right\} (\xi f) \xi_{(ij)}
\end{aligned}$$

$$C_{ij} = -\frac{1}{30\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+2)!}{2N!} f^{0N} + \sum_{n=1} C_1(N, n) f^{0N-2n} (ff)^n \right\} \xi^{mn} f_{(mn} f_{ij)}$$

$$\begin{aligned}
& -\frac{8}{3\lambda^2} \sum_{N=0}^{\infty} \left\{ f^{0N} + \sum_{n=1} C_2(N, n) f^{0N-2n} (ff)^n \right\} \xi_{(ij)} \\
& + \frac{1}{35\lambda^2} \sum_{N=0}^{\infty} \left\{ f^{0N} + \sum_{n=1} C_3(N, n) f^{0N-2n} (ff)^n \right\} \xi^{mn} f^{rs} f_{(mn} f_{rs} f_{ij)} \\
& + \frac{1}{9\lambda^2} \sum_{N=0}^{\infty} \left\{ f^{0N} + \sum_{n=1} C_4(N, n) f^{0N-2n} (ff)^n \right\} (\xi f) f_{(ij)}
\end{aligned}$$

$$\begin{aligned}
D = & -\frac{1}{15\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+3)!}{N!} f^{0N} + \sum_{n=1} D_1(N, n) f^{0N-2n} (ff)^n \right\} \xi^{(ij} f^{kl)} f_{(ij} f_{kl)} \\
& -\frac{8}{3\lambda^2} \sum_{N=0}^{\infty} \left\{ (N+1) f^{0N} + \sum_{n=1} D_2(N, n) f^{0N-2n} (ff)^n \right\} (\xi f)
\end{aligned}$$

$$\begin{aligned}
E = & -\frac{8}{3\lambda^2} \sum_{N=0}^{\infty} \left\{ (N+3) f^{0N} + \sum_{n=1} E_1(N, n) f^{0N-2n} (ff)^n \right\} \\
& + \frac{8}{9\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+3)!}{6N!} f^{0N} + \sum_{n=1} E_2(N, n) f^{0N-2n} (ff)^n \right\} (\xi f)^2 \\
& + \frac{4}{45\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+5)!}{12N!} f^{0N} + \sum_{n=1} E_3(N, n) f^{0N-2n} (ff)^n \right\} (\xi f) \xi^{(ij} f^{kl)} f_{(ij} f_{kl)}
\end{aligned}$$

$$\begin{aligned}
F = & +\frac{8}{\lambda^2} \sum_{N=0}^{\infty} \left\{ (N+1) f^{0N} + \sum_{n=1} F_1(N, n) f^{0N-2n} (ff)^n \right\} \\
& -\frac{1}{63\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+7)!}{6!N!} f^{0N} + \sum_{n=1} F_2(N, n) f^{0N-2n} (ff)^n \right\} \xi^{(ij} \xi^{kl} f^{mn} f^{rs)} f_{(ij} f_{kl} f_{mn} f_{rs)} \\
& -\frac{1}{45\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+5)!}{3N!} f^{0N} + \sum_{n=1} F_3(N, n) f^{0N-2n} (ff)^n \right\} (\xi f) \xi^{(ij} f^{kl)} f_{(ij} f_{kl)} \\
& -\frac{4}{5\lambda^2} \sum_{N=0}^{\infty} \left\{ \frac{(N+3)!}{3N!} f^{0N} + \sum_{n=1} F_4(N, n) f^{0N-2n} (ff)^n \right\} (\xi f)^2
\end{aligned}$$

We precise here below the convention notations used in writing our derived metric.

a) In writing the formulas, we have used the following notation

$$(f.f) = f^{(nm)} f_{(nm)}$$

b) The coefficients $A_i(N, n), B_i(N, n), \dots F_i(N, n)$ are finite numerical values defined for $n \geq 1$ and $N \geq 2n$. Simple examples are given by

$$\begin{aligned} A_1(2, 1) &= -\frac{5}{6} \\ B_1(2, 1) &= -\frac{4}{6} \\ C_1(2, 1) &= -\frac{1}{6} \\ D_1(2, 1) &= -\frac{5}{6} \\ E_1(2, 1) &= -\frac{3}{6} \\ F_1(2, 1) &= -\frac{3}{6} \\ &\vdots \end{aligned}$$

c) One can easily read this metric by remarking the fact that in each term of the components $A_{ijkl}, B_{ij}, C_{ij}, D, E$ and F , we have a general coefficient term given by

$$K(\star, N, n, f) = (\alpha(\star)) f^{0N} + \sum_{n=1} (\beta(\star)) f^{0N-2n} (ff)^n$$

where (\star) and $\alpha(\star), \beta(\star)$ denote respectively, the components $A_{ijkl}, B_{ij}, C_{ij}, D, E, F$, and their associate coefficients in the direction of f^{0N} and $f^{0N-2n} (ff)^n$. As an example we consider

$$K(A_{ijkl}, N, n, f) = \left(\frac{N+3}{2N!}\right) f^{0N} + \sum_{n=1} A_1(N, n) f^{0N-2n} (ff)^n$$

We guess that the terms $K(\star, N, n, f)$ described previously, should describe some general term of mathematical series which can be very useful to simplify more the obtained metric.

d) We point out that a simple choice can be done for $\lambda = 1$ at the level of Eq.(3.20) which reduce to

$$S = \int d^2z du \partial_{++} \partial_{--} \omega \{8 - 2\lambda \xi^{--2} \partial^{++2} \omega\}$$

This choice correspond simply to cancel in Eq.(4.2) the terms proportional to ξ^{--} .

REFERENCES

- [1] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, JETP Lett. 40 (1984) 912
A. Galperin, E. Ivanov, S. Lalitzin, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. 1 (1984) 469
- [2] A. Galperin, E. Ivanov, V. Ogievetsky and P.K. Townsend Class. Quant. Grav. 3 (1984) 625
A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, Comm. Math. Phys. 103 (1986) 515-526
- [3] L. Alvarez Gaumé and D.Z. Freedman, Comm. Math. Phys. 80 (1981) 443
- [4] N.J. Hitchin, A. Karlhede, U. Lindstrom and M. Rocek, Comm. Math. Phys. 108 (1987) 535
- [5] G.W. Gibbons and N.S. Manton, Phys. Lett. B 356 (1995) 32
G.W. Gibbons and P. Rychenkova, hep-th/9608085
- [6] M. Atiyah and N.Hitchin, Phys. Lett. A107(1985)21
G. Gibbons and N. Manton, Nucl. Phys. B274(1986)183
- [7] P. How, K. Stelle and P.K. Townsend Nucl. Phys. B214 (1983) 319
- [8] A. Galperin, E. Ivanov, V. Ogievetsky Nucl. Phys. B214 (1987) 74
- [9] U U. Lindstrom, M. Rocek Nucl. Phys. B222 (1983) 285
- [10] E.H. Saidi and M.B. Sedra Mod. Phys. Lett. AV9N34 (1994) 3163
- [11] A. Leznov and M. Saviliev, Lett. Math. Phys. 3(1979) 489, Comm. Math. Phys. 74(1980)111 P. Mansfield, Nucl.Phys. B208 (1982) 277.
- [12] E. Witten, hep-th/9703166